The Decay of Surface Waves produced by a Superposed Layer of Viscous Fluid.

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§ 1. In a paper published recently by Prof. R. W. Wood, on "The Mercury Paraboloid as a Reflecting Telescope,"* he notes the complete damping of ripples in the mercury surface produced by a thin superposed layer of glycerine. It has been suggested to me by Prof. Larmor that an interesting approximate estimate of the effect to be expected might be obtained from the results in my paper "The Influence of Viscosity on the Oscillations of Superposed Fluids."† In the case of a film of oil on the surface of water, in which the physical circumstances are different on account of the extreme tenuity of the film, Prof. Lamb has made an approximate calculation of the effect in his Treatise on Hydrodynamics,‡ by supposing the surface to be inextensible for small wave-lengths.

In my paper referred to, the approximations were obtained on the supposition that the viscosity in both liquids was small—the amplitude of the waves being as usual very small. In the present case, although the viscosity in mercury is very small, that in glycerine is comparatively large. Hence a new approximation is now required.

§ 2. The characteristics of the wave-motion at the free surface of a single liquid of great depth have been obtained for the two cases of small and of great viscosity respectively. In the second case there are two modes of vibration, both being dead-beat, for which the moduli of decay are $2k\nu/g$ and $(0{\cdot}91\nu k^2)^{-1}$, respectively.§ For application to glycerine the viscosity is not sufficiently great to make the approximation valid, except for wave-lengths less than 1 cm. I have thought it instructive to give this case a special consideration, although it is a side issue, as it affords an example of a dead-beat type of vibration changing to a propagated type as the wave-length is increased.

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* 'The Astrophys. Journ.,' vol. 29, No. 2.

† 'London Math. Soc. Proc.,' Ser. 2, vol. 6, p. 396.

‡ 3rd Edition, p. 571.

§ Lamb, 'Hydrodynamics,' 3rd Edition, p. 567.

For the type of wave-motion with which we are concerned, the velocity system (u, v) is given by

$$u = \frac{\partial\phi}{\partial x}+\frac{\partial\psi}{\partial y}, \qquad v = \frac{\partial\phi}{\partial y}-\frac{\partial\psi}{\partial x},$$

where

$$\phi = \mathrm{A}e^{ky}e^{\iota kx+\alpha t},$$
$$\psi = \mathrm{H}e^{\lambda y}e^{\iota kx+\alpha t},$$
$$\lambda^2 = k^2+\alpha/\nu\,;$$

ν is the kinematical coefficient of viscosity and is equal to μ/ρ.

Writing $\alpha/k^2\nu = \tan^2\theta$, the period equation can be put into the form

$$\tan^4\theta+4\tan^2\theta+g/k^3\nu^2 = 4(\sec\theta-1).^* \tag{1}$$

When ν is great the term $g/k^3\nu^2$ can be neglected in the determination of one mode, and the equation reduces to

$$\sec^3\theta+\sec^2\theta+3\sec\theta-1 = 0. \tag{2}$$

We obtain an admissible root $\sec\theta = 0{\cdot}3$, whence $\alpha = -0{\cdot}91\nu k^2$.

For glycerine, $\nu = 9{\cdot}64$, hence the term neglected above must be retained when the wave-length is greater than 1 cm.

Writing $y = \sec\theta$, the equation (1) becomes

$$y^4+2y^2-4y+1+g/k^3\nu^2 = 0. \tag{3}$$

It can be solved by Descartes' method, which leads to the cubic

$$\mathrm{K}^6+4\mathrm{K}^4-4g\mathrm{K}^2/k^3\nu^2-16 = 0\ . \tag{4}$$

The two values of y, which are admissible, are obtained from the equation

$$y^2-\mathrm{K}y+(1+\tfrac{1}{2}\mathrm{K}^2-2/\mathrm{K}) = 0, \tag{5}$$

where K^2 is the positive root of (4), and K itself is taken positively.

It will be seen from the table given below that the rate of decay is always great, but that waves of length greater than a certain limit, which lies between 2 and 3 cms., are propagated.

When the wave-length is very great, one root of (4) is given by

$$\mathrm{K}^2 = -2+s^{\frac{1}{2}}, \quad \text{where} \quad s = 4g/k^3\nu^2\,;$$

whence

$$y^2-\{-2+s^{\frac{1}{2}}\}^{\frac{1}{2}}\,y+\tfrac{1}{2}s^{\frac{1}{2}} = 0.$$

We obtain

$$\alpha/k^2\nu = -2\pm\tfrac{1}{2}is^{\frac{1}{2}},$$

or

$$a = -2k^2\nu\pm i\sqrt{gk}.$$

Thus α has the same form when νk^2 is small as we obtain in the case of a slightly viscous liquid.

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* *Cf.* Basset, 'Hydrodynamics,' vol. 2, p. 312.

Decay of Wave-motion on Glycerine.

L.	sec θ.	$\alpha/k^2\nu$.	τ.	v.
1	$0{\cdot}3$	$-0{\cdot}91$	$0{\cdot}003$	
	$0{\cdot}99$	$-0{\cdot}02$	$0{\cdot}131$	
2	$0{\cdot}43$	$-0{\cdot}815$	$0{\cdot}013$	
	$0{\cdot}89$	$-0{\cdot}208$	$0{\cdot}051$	
3	$0{\cdot}715 \pm 0{\cdot}347i$	$-0{\cdot}61 \pm 0{\cdot}496i$	$0{\cdot}038$	$10{\cdot}0$
5	$0{\cdot}92 \pm 0{\cdot}877i$	$-0{\cdot}92 \pm 1{\cdot}61i$	$0{\cdot}071$	$19{\cdot}6$
10	$1{\cdot}65 \pm 1{\cdot}45i$	$-1{\cdot}40 \pm 6{\cdot}05i$	$0{\cdot}188$	$36{\cdot}7$
20	$2{\cdot}95 \pm 3{\cdot}06i$	$-1{\cdot}67 \pm 18{\cdot}0i$	$0{\cdot}629$	$52{\cdot}9$
30	$4{\cdot}06 \pm 4{\cdot}18i$	$-1{\cdot}96 \pm 38{\cdot}1i$	$1{\cdot}18$	$76{\cdot}9$

L is the wave-length in cms. τ the modulus of decay in seconds.
v the velocity of propagation in cms. per second.

§ 3. In the general problem we shall consider the motion of two fluids, both of depth large compared with the wave-length of the motion, in the two cases: (1) both very viscous; (2) the lower one slightly viscous, the upper very viscous.

The period equation for waves at the interface between a liquid of density ρ and viscosity ν, and a superposed liquid of density ρ' and viscosity ν', is*

$$\begin{aligned}&4k^3(\nu\rho-\nu'\rho')^2(k-\lambda)(k-\lambda')+4k^2\alpha(\nu\rho-\nu'\rho')[\rho(k-\lambda')-\rho'(k-\lambda)]\\&+\rho^2(\alpha^2+gk)(k-\lambda')+\rho'^2(\alpha^2-gk)(k-\lambda)\\&-\rho\rho'[2k\alpha^2+\alpha^2(\lambda+\lambda')+gk(\lambda-\lambda')]=0,\end{aligned}$$

where $$\lambda^2=k^2+\frac{\alpha}{\nu},\qquad \lambda'^2=k^2+\frac{\alpha}{\nu'}.$$

Viscosities both Large.

§ 4. (*a*) For the mode in which α is large, assuming that α is of the order ν or ν', the equation determining α is

$$\begin{aligned}&4k^3(\nu\rho-\nu'\rho')^2(k-\lambda)(k-\lambda')+4k^2\alpha(\nu\rho-\nu'\rho')[\rho(k-\lambda')-\rho'(k-\lambda)]\\&+[\rho^2(k-\lambda')+\rho'^2(k-\lambda)-\rho\rho'(2k^2+\lambda+\lambda')]\alpha^2=0.\end{aligned}$$

The analysis is much simplified by putting $\nu=\nu'$. With this assumption, after writing $\alpha=\nu k^2\tan^2\theta$, we derive the equation

$$(\rho+\rho')^2\sec^3\theta+(\rho^2-6\rho\rho'+\rho'^2)\sec^2\theta+(3\rho^2-2\rho\rho'+3\rho'^2)\sec\theta-(\rho-\rho')^2=0. \tag{6}$$

With this may be compared the equation (2) for a single liquid.

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* 'London Math. Soc. Proc.,' Ser. 2, vol. 6, p. 398. I have quoted it again at length as there are two misprints as it has been previously printed.

Some numerical solutions of (6) are given in the table below:—

ρ'/ρ …………	0	0·1	0·5	1
sec θ ………	0·29	0·27	0·097	0
$\alpha/\nu k^2$ ………	−0·91	−0·93	−0·98	−1

When ν and ν' are different, we can safely say that the modulus of decay is not greater than $(0{\cdot}91\nu_1 k^2)^{-1}$, where ν_1 is the smaller of the two coefficients.

(*b*) For the mode in which α is small, we assume that α is of the order ν^{-1}. We easily find that

$$\alpha = -\frac{g(\rho-\rho')}{2k(\rho\nu+\rho'\nu')}. \tag{7}$$

This corresponds to $\alpha = -g/2k\nu$ for a single liquid.

Upper Liquid Viscous, Lower Slightly Viscous.

§ 5. (*a*) For the mode in which α is large, writing $\alpha/k^2\nu' = \tan^2\theta$ in the terms of highest importance, we obtain the equation (*cf.* 6)

$$(\rho+\rho')\sec^3\theta + (\rho+\rho')\sec^2\theta + (3\rho'-\rho)\sec\theta - (\rho+\rho') = 0. \tag{8}$$

The nature of the solution of this equation is shown by the following table:—

ρ'/ρ …………	0·1	0·5	0·9
sec θ ………	0·92	0·675	0·565
$\alpha/\nu k^2$ ………	−0·15	−0·544	−0·68

(*b*) For the mode in which α is small we have

$$\alpha = -\frac{g(\rho-\rho')}{2k\nu'\rho'}. \tag{9}$$

This can be obtained from (7) by putting $\nu = 0$.

§ 6. *The Case of a Thin Layer of Glycerine superposed on Deep Mercury.*—The period equation for this case is given in § 4 of my paper (in the first term of the fourth column ρ should be replaced by ρ'). The effect of capillarity can be included by writing $g+T_2k^2/\rho'$ instead of g in the second column, and $g+T_1k^2/\rho$ instead of g in the fifth and sixth terms of the fourth column, where T_1 is the surface tension of the glycerine-mercury interface, and T_2 is the surface tension of the glycerine free surface.

For glycerine at 17°, C. $\nu' = 9{\cdot}64$, and for mercury, $\nu = 0{\cdot}0011$, consequently we shall approximate by putting $\nu = 0$ at once. This greatly simplifies the period equation, and does not affect its validity except when the modulus of decay is very large. It will be further assumed that kh and $\lambda' h$ are small, so that squares of these quantities will be neglected.

With these assumptions the period equation becomes

$$\begin{aligned}&(\rho+\rho' kh)\,\alpha^4+4k^2\nu'\,(\rho+\rho' kh)\,\alpha^3\\&+[g\rho\,(1+kh)+k^2\,(\mathrm{T}_1+\mathrm{T}_2)+k^3h\mathrm{T}_2\,\rho/\rho']\,k\alpha^2\\&+[g\rho+k^2\,(\mathrm{T}_1+\mathrm{T}_2)]\,4k^3\nu'\alpha\\&+k^3h\,(g+k^2\mathrm{T}_2/\rho')\,[(\rho-\rho')\,g+k^2\mathrm{T}_1] = 0.\end{aligned}$$

For wave-lengths less than a certain limit this equation has two negative roots and two complex ones. The two complex roots correspond to the mode in which the two liquids vibrate as a single liquid without slip at their common surface. The two negative roots show that the waves of the other mode are dead-beat.

It appears from the numerical calculation, in the case of $h = 1$ mm., that very approximately the two negative roots are obtained from the last two and the first two terms of the equation respectively. Thus they are

$$\alpha_1 = -\frac{(g+k^2\mathrm{T}_2/\rho')\,[g\,(\rho-\rho')+k^2\mathrm{T}_1]}{4\nu'\,[k^2\,(\mathrm{T}_1+\mathrm{T}_2)+g\rho]},$$

$$\alpha_2 = -4k^2\nu'.$$

These approximations are not, however, sufficiently good to be employed in the determination of the two remaining roots, as they necessarily make the real part of these roots to be positive, equal to $-\tfrac{1}{2}\alpha_1$.

For wave-lengths greater than the limit mentioned above all the roots are complex, and all the waves are propagated. This limit is given, in general, by the condition for equality of the roots of the last three terms of the equation equated to zero.

For a glycerine-air surface, $\mathrm{T} = 65$; for a mercury-air surface, $\mathrm{T} = 540$. It is assumed for the purpose of an illustrative calculation that $\mathrm{T}_1 = 300$ at the common surface, while $\mathrm{T}_2 = 65$. In the table below the effect is exhibited of a layer of glycerine of 1 mm. depth on the modulus of decay and velocity of propagation of wave-motion at the surface of mercury.

The damping effect of a thin layer of glycerine is very noticeable from the table. The rate of decay of any arbitrary disturbance will depend, in general, on the rates of decay of those of the modes, into which it can be analysed, which are propagated. The other modes disappear instantaneously. As a consequence of this the curvature of the upper glycerine surface is always the same as that of the mercury surface, a fact which may be of importance in the application to an optical instrument such as the one which gave rise to this investigation. Prof. Wood made use of a depth of 4 mm. of glycerine; a layer of this depth would give a much greater rate of decay.

L.	τ.	τ_c.	τ_0.	v.	v_c.	v_0.
2	0·435	0·291				
	0·0026*	0·0027*	45·5			20·7
	10·87	5·69		17·5	19·7	
5	0·417	0·402				
	0·017	0·017	303			28·8
	12·99	10·03		28·1	28·9	
10	0·352	0·354				
	0·081	0·082	1100			39·8
	39·95	27·03		39·5	40·0	
20	0·518	0·518	5000	7·19	7·29	54·1
	†	†		54·1	54·1	

L is the wave-length in cms.

τ is the modulus of decay when capillarity is neglected; τ_c that when capillarity is included; τ_0 that of waves at the surface of clean mercury (T = 540).

v, v_c, v_0 are the velocities of propagation in the corresponding cases.

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* In the case of these quantities the approximation, on the supposition that $\lambda'h$ is small, breaks down; but the rate of decay is in any case large.

† The rate of decay cannot be determined without great labour.

If the wave-length is small compared with the depth, the rates of decay can be derived from (8) and (9). These approximations are applicable to glycerine, as the wave-length is small. The moduli of decay for the two modes are $(1\cdot3k^2)^{-1}$ and $(491k)^{-1}$; both of these are extremely small.

In these investigations the damping effect of the air has been neglected. That this is very considerable in the case of water waves has been shown in my paper, to which reference has been made; but owing to the great density of mercury its effect will be small, especially for small wave-lengths, such as those tabulated. Its effect in the case of dead-beat modes is absolutely negligible.

It may be mentioned that the mercury used by Prof. Wood was of comparatively small depth; this would cause considerable damping in the case of the longer waves.

[*Note added June* 19.—It will be noticed from the foregoing table that the modulus of decay, when the vibrations are influenced by the glycerine layer, increases much more slowly with the wave-length than in the case of a clean mercury surface. This fact may go a little way towards an explanation of the damping of waves on water by oil. But a complete explanation is to be looked for in a totally different direction. Dr. Houstoun notes an experimental result which is apparently contradictory to usual experience.*

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* 'Phil. Mag.,' January, 1909.

The modulus of decay (τ) for waves of length 5 cms. when the surface of the mercury is covered by 4 mm. of glycerine is 4·9 secs. All modes become propagated for wave-lengths greater than some value between 5 and 10 cms. For a depth of 1 mm. of glycerine this limit lies between 10 and 20 cms., and thus we see that as the depth is increased this critical value of the wave-length approaches that for waves at the surface of deep glycerine, which lies between 2 and 3 cms.]

The Elastic Limits of Iron and Steel under Cyclical Variations of Stress.

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(Abstract.)

An explanation of fatigue is developed in the paper which is in accordance with all the researches on the fracture of materials by the cyclical repetition of stress. The theory was put forward by Bauschinger in 1886, when he suggested that the necessary condition of safety was that the repeated stresses applied should be within the limits of elasticity of the specimen, and that the least variation from this condition introduces fatigue and ultimately fracture occurs.

In order for this to be true for Wöhler's well-known experiments, the elastic limits must be variable within very wide limits, and the present paper describes observations made during the adjustment of the limits of elasticity to any particular condition of experiment.

It is now found that iron or steel is capable of adjusting itself to variations of stress, cyclically applied, after a sufficient number of repetitions. When the adjustment is complete, the specimen under test is found to have become perfectly elastic throughout the whole cycle, and fatigue does not occur.

During the adjustment of the elastic limits to a given cycle of stress, a change of length occurs in the specimen, which corresponds to the extension observed in an ordinary tensile test when the yield stress is exceeded. For stress cyclically applied, this extension occurs even when the maximum stress in the cycle is less than the static yield stress.